John Barnard
Steven Lund
USPAS
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I. Introduction

(related reading in parentheses)

Particle motion (Reiser 2.1)
Equation of motion (Reiser 2.1)
Dimensionless quantities (Reiser 4.2)

Plasma physics of beams (Reiser 3.2, 4.1)

Emittance and brightness (Reiser 3.1 - 3.2)

PARTICLE EQUATIONS OF MOTION / DIMENSIONLESS QUANTITIES

CONSIDER THE LORENTZ FORCE ON A PARTICLE UNDER THE INFLUENCE OF ELECTRIC AND MAGNETIC FORCES.

$$\frac{d\underline{p}}{dt} = q(\underline{E} + \underline{v} \times \underline{B}) \qquad [\text{SI units}]$$

$$\underline{p} = \gamma m \underline{v} \qquad \gamma^2 = \frac{1}{1-\beta^2} \qquad \beta = \frac{v}{c}$$

CONSIDER THE X-COMPONENT OF THE MOTION (TRANSVERSE TO THE STREAMING MOTION OF THE PARTICLE)

TRANSFORM TO S AS THE INDEPENDENT VARIABLE.

$$dt = \frac{ds}{v_z} \quad \Rightarrow \quad v_x = \frac{dx}{dt} = v_z x' \qquad ' = \frac{d}{ds}$$

$$m v_z \frac{d}{ds}(\gamma v_z x') = q(\underline{E} + \underline{v} \times \underline{B})_x$$

$$\gamma m v_z^2 x'' + x' m v_z \frac{d}{ds}(\gamma v_z) = q(\underline{E} + \underline{v} \times \underline{B})_x$$

$$\Rightarrow \quad x'' + \left[\frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)\right] x' = \frac{q}{\gamma m v_z^2}(\underline{E} + \underline{v} \times \underline{B})_x$$

NOW CONSIDER AN UNBUNCHED BEAM OF UNIFORM DENSITY ρ AND CIRCULAR CROSS SECTION

$$\underline{\nabla}\cdot\underline{E} = \frac{\rho}{\epsilon_0}$$

$$2\pi r E_r = \frac{\pi r^2 \rho}{\epsilon_0} \quad \text{(Gauss' theorem)}$$

$$\lambda = \pi r_b^2 \rho$$

$$E_r = \frac{\rho}{2\epsilon_0} r = \frac{\lambda}{2\pi\epsilon_0}\frac{r}{r_b^2} \qquad E_x = E_r\cos\theta = E_r\left(\frac{x}{r}\right) = \frac{\lambda}{2\pi\epsilon_0}\frac{x}{r_b^2}$$

Similarly $\underline{\nabla}\times\underline{B} = \mu_0 \underline{J}$

$$2\pi r B_\theta = \mu_0 \rho v_z \pi r^2 \quad \text{(Stokes theorem)}$$

$$B_\theta = \frac{\mu_0 \lambda v_z}{2\pi}\frac{r}{r_b^2} \qquad B_y = \frac{\mu_0 \lambda v_z}{2\pi}\frac{x}{r_b^2} \qquad (B_z = 0)$$

Let $(\underline{E} + \underline{v}\times\underline{B})_x = (E_x - v_z B_y)^{self} + (E_x + v_y B_z - v_z B_y)^{ext}$

$$\Rightarrow \quad x'' + \left[\frac{1}{\gamma v_z}\frac{d}{ds}\gamma v_z\right]x' = \frac{q}{\gamma m v_z^2}\frac{\lambda}{2\pi\epsilon_0}\frac{x}{r_b^2}\left[1 - \mu_0\epsilon_0 v_z^2\right] + \frac{q}{\gamma m v_z^2}(\underline{E}+\underline{v}\times\underline{B})_x^{ext}$$

USING $\mu_0\epsilon_0 = \frac{1}{c^2}$

Assuming $\beta_x^2 + \beta_y^2 << \frac{1}{\gamma^2} \Rightarrow \gamma^2 \simeq \frac{1}{1 - v_z^2/c^2}$ (PARAXIAL APPROXIMATION)

($\Rightarrow \tilde{\beta}_x^2 + \tilde{\beta}_y^2 << 1$, HERE ~ INDICATES VALUE IN COMOVING FRAME [NON-RELATIVISTIC MOTION IN COMOVING FRAME].)

LUMPING EXTERNAL FORCE INTO A LINEAR FIELD:

$$x'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)\,x' \simeq \frac{q}{\gamma^3 m v_z^2}\,\frac{\lambda}{2\pi\epsilon_0}\,\frac{x}{r_b^2} - \underbrace{K(z)\,x}_{\text{EXTERNAL FORCES}}$$

$$\equiv Q\,\frac{x}{r_b^2} - K(z)\,x$$

$$Q = \frac{q\lambda}{2\pi\epsilon_0 \gamma^3 m v_z^2} = \text{GENERALIZED PERVEANCE} \longrightarrow \begin{cases} \dfrac{\lambda}{4\pi\epsilon_0 V} & \text{for } \gamma^2 v_z^2 << c^2 \\[2ex] \dfrac{\lambda}{2\pi\epsilon_0 V\left(\frac{qV}{mc^2}\right)^2} & \text{for } \gamma^2 v_z^2 >> c^2 \end{cases}$$

$$= \frac{(q/e)}{(m/m_{amu})}\,\frac{2I}{I_0}\,\frac{1}{\gamma^3\beta^3} \qquad I_0 = \frac{4\pi\epsilon_0 m_{amu} c^3}{e} \cong 31\ \text{MA}$$

here $qV \equiv (\gamma - 1)\,mc^2$

Also note in non-relativistic limit $Q = \left(\frac{m}{2q}\right)^{1/2} \frac{1}{4\pi\epsilon_0 c}\left(\frac{I}{V^{3/2}}\right)$

(same scaling as original term perveance characterizing in pierces).

$$Q \simeq \frac{\Phi_{SELF}}{V} = \frac{\int_0^{r_b}(E_r - v_z B_\theta)\,dr}{V} = \frac{\text{POTENTIAL ENERGY OF BEAM PARTICLE}}{\text{KINETIC ENERGY OF " "}}$$

SOMETIMES PERIODIC FOCUSING IS EMPLOYED

$$K(z) = K(z + S)$$

$$S = \text{PERIOD}$$

FOR SOME PURPOSES A SUITABLE CONSTANT CAN BE FOUND WHICH CAPTURES SLOW VARIATION OF THE PARTICLE MOTION. (SMOOTH FOCUSING APPROX.)

$$\Rightarrow \boxed{x'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)\,x' = Q\,\frac{x}{r_b^2} - k_{\beta 0}^2\,x}$$

$k_{\beta 0} \equiv$ "UNDEPRESSED BETATRON FREQUENCY" $\qquad \sigma_0 \equiv k_{\beta 0} S$ = UNDEPRESSED PHASE ADVANCE

If $\frac{dV_z}{dz} = 0$ [drifting beams]

$$x'' = -\left[k_{\beta 0}^2 - \frac{Q}{r_b^2}\right] x$$

$$= -k_{\beta 0}^2 \underbrace{\left[1 - \frac{Q}{k_{\beta 0}^2 r_b^2}\right]}_{= \left(\frac{\sigma}{\sigma_0}\right)^2 = (\text{"TUNE DEPRESSION"})^2} x \quad = -k_\beta^2 x$$

"DEPRESSED BETATRON FREQUENCY"

EFFECT OF SPACE CHARGE IS TO LOWER FREQUENCY OF HARMONIC OSCILLATIONS

$\frac{\sigma}{\sigma_0} = 0 \Rightarrow$ FULLY TUNE DEPRESSED

$\frac{\sigma}{\sigma_0} = 1 \Rightarrow$ NO SPACE-CHARGE DEPRESSION

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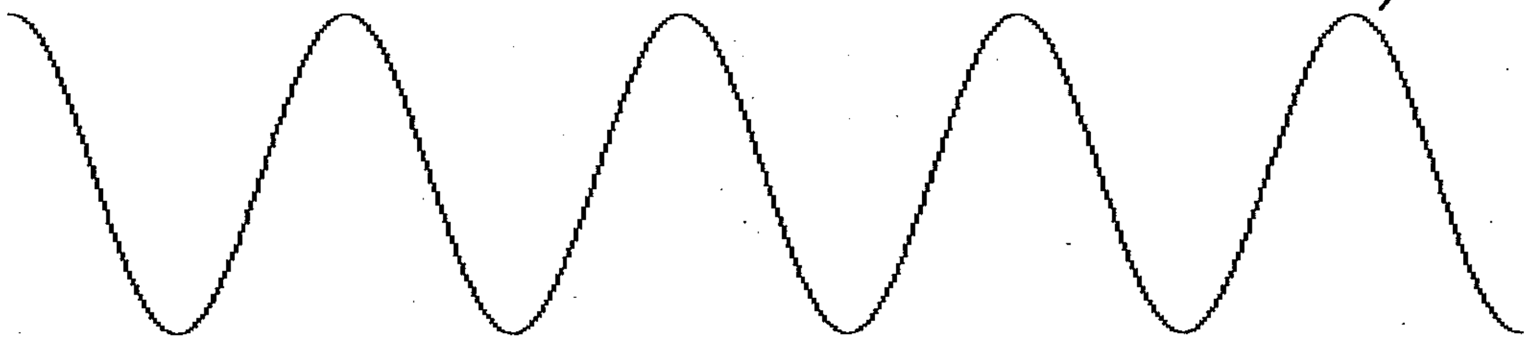
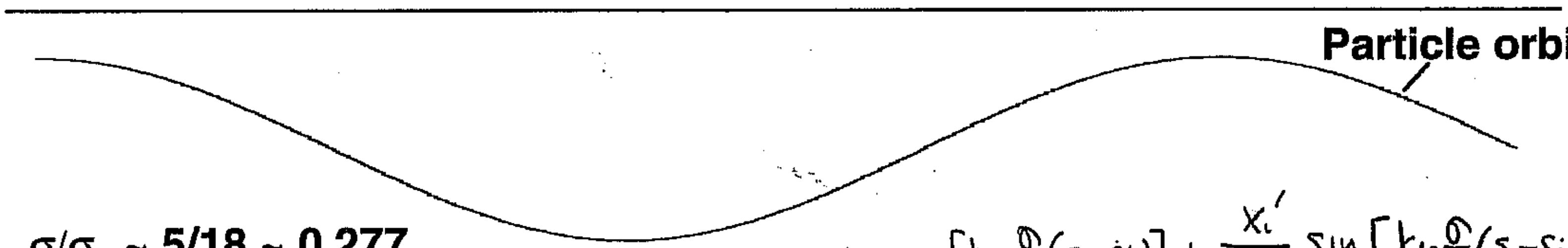
Space charge reduces betatron phase advance

Without space charge: $X = X_i \cos\left[k_{\beta 0}(s-s_i)\right] + \frac{X_i'}{k_{\beta 0}} \sin\left[k_{\beta 0}(s-s_i)\right]$



With space charge:



σ/σ_0 ~ **5/18 ~ 0.277**

$X = X_i \cos\left[k_{\beta 0}\frac{\sigma}{\sigma_0}(s-s_i)\right] + \frac{X_i'}{\left(\frac{\sigma}{\sigma_0}\right)k_{\beta 0}} \sin\left[k_{\beta 0}\frac{\sigma}{\sigma_0}(s-s_i)\right]$

BENDING BEAMS

RETURNING TO PARTICLE EQUATION WITH ARBITRARY E, & B:

$$x'' + \left[\frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)\right] x' = \frac{q}{\gamma m v_z^2}\left(\underline{E} + \underline{v}\times\underline{B}\right)_x$$

IF EXTERNAL FORCE IS PROPORTIONAL TO $-x$

$\Rightarrow$ FOCUSING (HARMONIC OSCILLATIONS)

HOWEVER, IF $\underline{E} + \underline{v}\times\underline{B}$ = CONSTANT

$\Rightarrow$ BENDING

EXAMPLE: If $B = B_y \hat{e}_y$

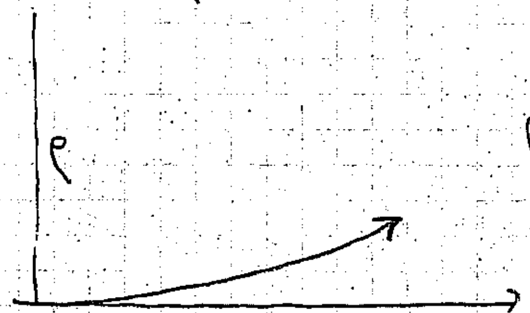
$\bar{v} = v_0 \hat{e}_z + v_x \hat{e}_x$ where $v_0 \gg v_x$

$$\Rightarrow \quad x'' = \frac{q B_y}{\gamma m v_z} = \frac{B_y}{[B\rho]}$$

$[B\rho] \equiv$ RIGIDITY $= \dfrac{\gamma m v_z}{q} = \dfrac{p}{q}$

$$x' = \frac{B_y}{[B\rho]} z + x_0'$$

$$x = \frac{B_y}{[B\rho]}\frac{z^2}{2} + x_0' z + x_0$$



ρ = RADIUS OF CURVATURE OF ARC $= \dfrac{[B\rho]}{B_y}$

(BENDING CAN ALSO BE CARRIED OUT WITH ELECTRIC FIELDS E_x = const).

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PLASMA PHYSICS OF BEAMS

PHYSICS OF SPACE-CHARGE ≡ PHYSICS OF SELF-FIELDS

≡ PLASMA PHYSICS OF PARTICLE BEAMS

PLASMA PARAMETER

$$q\Phi_{IP} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r}$$

AVERAGE POTENTIAL ENERGY $q\Phi_{IP}$ OF PARTICLE DUE TO ITS NEAREST NEIGHBOR (q = charge of particle)

$$\simeq \frac{1}{4\pi\epsilon_0} n_0^{1/3} q^2$$

If $q\Phi_{IP} << k_B T$ ⇒ PLASMA (weakly coupled plasma)

DEFINE $\lambda_D = \frac{(kT/m)^{1/2}}{(n_0 q^2/\epsilon_0 m)^{1/2}} = \frac{v_{TH}}{\omega_p} = \left(\frac{kT\epsilon_0}{n_0 q^2}\right)^{1/2}$ = DEBYE LENGTH

= SHIELDING DISTANCE EVEN IN NON-NEUTRAL PLASMA

DEFINE $\Lambda = \frac{4\pi}{3} n_0 \lambda_D^3$ ≡ PLASMA PARAMETER

$$\sim \left(\frac{k_B T}{q\Phi_{IP}}\right)^{3/2} >> 1$$

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REF. "INTRO. TO PLASMA THEORY", D.R. NICHOLSON WILEY, 1983.

KLIMONTOVICH EQUATION

Let $N(\underline{x},\underline{v},t) = \sum_{i=1}^{N_0} \delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))$

N_0 particles; $\underline{X}_i$, $\underline{V}_i$ are position and velocity of ith particle

$\dot{\underline{X}}_i = \underline{V}_i \qquad m\dot{\underline{V}}_i = q\underline{E}^m[\underline{X}_i(t),t] + q[\underline{V}_i \times \underline{B}^m[\underline{X}_i(t),t]]$ (non-relativistic)

$N(x,v,t)$ = "density" of particles in phase space

$$\int N\, d^3x\, d^3v = N_0$$

Let $u = x - X_i(t)$

$\frac{\partial}{\partial x} f(u) = f'(u)$

$\frac{\partial}{\partial t} f(u) = f'(u)(-\dot{X}(t))$

$= -\dot{X}(t)\frac{\partial}{\partial x} f(u)$

Taking derivative:

$$\frac{\partial N}{\partial t}(x,v,t) = -\sum_{i=1}^{N} \dot{\underline{X}}_i(t)\cdot\nabla_x\left[\delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))\right]$$
$$-\sum_{i=1}^{N_0} \dot{\underline{V}}_i(t)\cdot\nabla_v\left[\delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))\right]$$

MAXWELL'S EQUATIONS:

$$\nabla\cdot\underline{E}^m = \overbrace{\left(\frac{1}{\epsilon_0}\right) q\int dv\, N(x,v,t)}^{\rho^m} \qquad \nabla\cdot\underline{B}^m = 0$$

$$\nabla\times\underline{E}^m = -\frac{\partial \underline{B}^m}{\partial t} \qquad \nabla\times\underline{B}^m = \mu_0 \underbrace{q\int d\underline{v}\,\underline{v}\, N(x,v,t)}_{J^m} + \frac{\partial E^m}{\partial t}$$

$$\Rightarrow \frac{\partial N}{\partial t}(x,v,t) = -\sum_{i=1}^{N_0} \underline{V}_i(t)\cdot\nabla_x\left[\delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))\right]$$
$$-\sum_{i=1}^{N_0}\left(\left(\frac{q}{m}\right)E^m + \left(\frac{q}{m}\right)\left[\underline{V}_i\times\underline{B}^m[X_i(t),t]\right]\right)\cdot\nabla_v\left[\delta(x-X_i)\delta(v-V_i(t))\right]$$

Note that $\underline{V}_i(t)\,\delta(\underline{v}-\underline{V}_i(t)) = \underline{v}\,\delta(\underline{v}-V_i(t))$ so,

∇_x = spatial gradient

∇_v = gradient w.r.t. 3 velocity variables

$$\Rightarrow \frac{\partial N}{\partial t}(x,\underline{v},t) = -\underline{v}\cdot\nabla_x\sum_{i=1}^{N_0}\delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))$$
$$-\left(\frac{q}{m}E^m(\underline{x},t) + \frac{q}{m}(\underline{v}\times\underline{B}^m(\underline{x},t))\right)\cdot\nabla_v\sum_{i=1}^{N_0}\delta(\underline{x}-\underline{X}_i(t))\delta(v-V_i(t))$$

$$\boxed{\frac{\partial N}{\partial t}(\underline{x}\underline{v}t) = -\underline{v}\cdot\nabla_x N(\underline{x},\underline{v},t) + \frac{q}{m}(E^m + v\times B^m)\cdot\nabla_v N(\underline{x},\underline{v},t)}$$

Klimontovich Equation

Total derivative along an orbit:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \left.\frac{dx}{dt}\right|_{\text{ORBIT}} \cdot \nabla_x + \left.\frac{dv}{dt}\right|_{\text{ORBIT}} \cdot \nabla_v$$

$$\Rightarrow \boxed{\frac{D}{Dt} N(\underline{x}, \underline{v}, t) = 0}$$

Note that $N = 0$ or ∞, nothing in between.

Let $$f(x, v, t) = \frac{\int^{\Delta x^3, \Delta v^3} N(\underline{x}, \underline{v}, t)\, d^3x\, d^3v}{\Delta x^3 \Delta v^3}$$

over some box in phase space. Δx & Δv are the size of box

$$\equiv \langle N(\underline{x}, \underline{v}, t) \rangle$$

Assume $n^{-1/3} \ll \Delta x \ll \lambda_D$ so that $f(x, v, t)$ is smooth function.

Then

$$N = f + \delta f \qquad f = \langle N \rangle \qquad \langle \delta f \rangle = 0$$
$$E^m = E + \delta E \qquad E = \langle E^m \rangle \qquad \langle \delta E \rangle = 0$$
$$B^m = B + \delta B \qquad B = \langle B^m \rangle \qquad \langle \delta B \rangle = 0$$

$$\Rightarrow \underbrace{\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla_x f + \frac{q}{m}(\underline{E} + \underline{v} \times \underline{B}) \cdot \nabla_v f}_{\text{"SMOOTHLY VARYING PART"}} = \underbrace{-\frac{q}{m}\langle (\delta \underline{E} + \underline{v} \times \delta \underline{B}) \cdot \nabla_v \delta f \rangle}_{\text{AVERAGE OF "SPIKEY" QUANTITIES}}$$

"DISCRETE PARTICLE EFFECTS" OR "COLLISIONS"

If collisions are neglected (set RHS to zero):

$\boxed{\text{Vlasov - EQUATION}}$

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla_x f + \frac{q}{m}(\underline{E} + \underline{v} \times \underline{B}) \cdot \nabla_v f = 0$$

$$\Rightarrow \boxed{\frac{Df}{Dt} = 0}$$

PHASE SPACE DENSITY CONSTANT ON TRAJECTORIES. (LIOUVILLE'S THEOREM)

THE RHS IS DUE TO COLLISIONS WITH NON-SMOOTH FIELDS:

VERY HEURISTICALLY

$$-\frac{q}{m}\left\langle\left(\delta\underline{E} + \underline{v}\times\delta\underline{B}\right)\cdot\nabla_v \delta f\right\rangle \sim \nu_c f$$

$$\nu_c \sim \sigma n v$$

$\sigma \sim \pi r_c^2$ where r_c is given by $kT \sim \dfrac{q^2}{4\pi\varepsilon_0 r_c}$

(for large angle collisions)

$$\Rightarrow \nu_c \sim \pi\left(\frac{q^2}{4\pi\varepsilon_0 kT}\right)^2 n_0\left(\frac{kT}{m}\right)^{1/2} \leftarrow$$

(VERY ROUGH, BUT MAIN SCALING IS CORRECT, WITH LOGARITHMIC CORRECTION FACTORS)

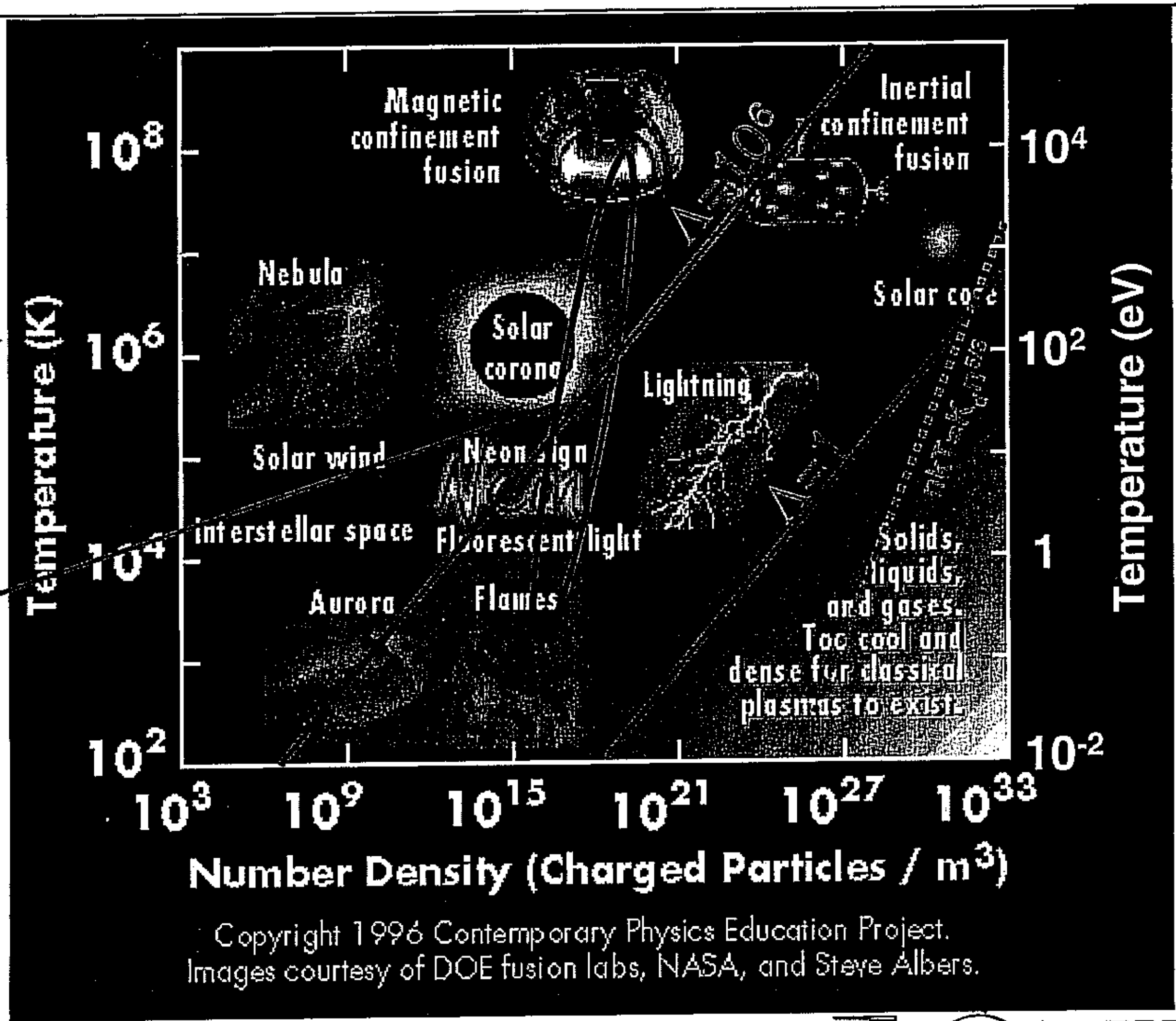
ON LHS of VLASOV EQUATION:

$$\frac{q}{m}E\,\nabla_v f \sim \left(\frac{q\lambda_D n_0}{\varepsilon_0}\right)\frac{f}{v_{TH}} \qquad \text{where } v_{TH}\sim\sqrt{\frac{kT}{m}}$$

$$\frac{\text{COLLISION TERM}}{\text{LHS}} \sim \frac{1}{16\,\lambda_D^3 n_0} = \frac{1}{16\Lambda}$$

Accelerator beams are non-neutral plasmas

Accelerator beams for Heavy Ion Fusion



DESCRIPTION OF THE BEAM

LIOUVILLE'S THEOREM: $\frac{df}{dt} = 0$ along a trajectory in phase space.

Let $dN = f\, dx\, dy\, dz\, dp_x\, dp_y\, dp_z$

The continuity equation in phase space is:

$$\frac{\partial f}{\partial t} + \underline{\nabla} \cdot (f\underline{v}) = 0$$

where $\underline{v} = \frac{d}{dt}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$ & $\underline{\nabla} \cdot \underline{a} = \frac{\partial a_1}{\partial q_1} + \frac{\partial a_2}{\partial q_2} + \frac{\partial a_3}{\partial q_3} + \frac{\partial a_{p_1}}{\partial p_1} + \frac{\partial a_{p_2}}{\partial p_2} + \frac{\partial a_{p_3}}{\partial p_3}$

($\underline{v}$ & $\underline{\nabla}\cdot$ are the 6-D velocity & divergence, respectively).

If the system is governed by a Hamiltonian $H(\underline{q}, \underline{p}, t)$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = \frac{-\partial H}{\partial q_i}$$

Now, $\underline{\nabla} \cdot \underline{v} = \sum_{i=1}^{3} \frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} = \sum_{i=1}^{3} \frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} = 0$

$$\Rightarrow \frac{\partial f}{\partial t} + f \underbrace{\underline{\nabla} \cdot \underline{v}}_{0} + \underline{v} \cdot \nabla f = 0$$

$$\Rightarrow \boxed{\left.\frac{df}{dt}\right|_{\substack{6D \\ \text{trajectory}}} = 0}$$

Emittance & Brightness

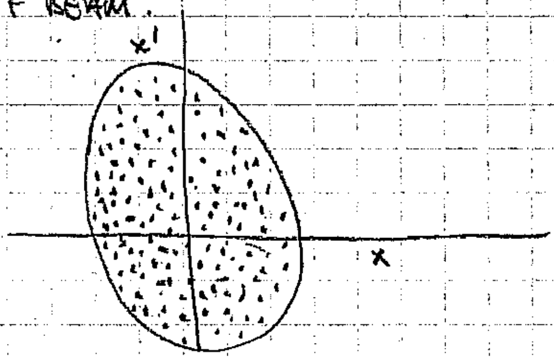
LIOUVILLE'S EQUATION OR VLASOV EQUATION $\Rightarrow \dfrac{dN}{dx\,dy\,dz\,dp_x\,dp_y\,dp_z} = \text{const}$

IF $x'' = f(x)$ AND NOT FUNCTIONS (y & z)
$y'' = f(y) =$ " " " (x, & z)
$z'' = f(z) =$ " " " (x, & y)

THEN $\dfrac{dN}{dx\,dp_x} = \text{const}$; $\dfrac{dN}{dy\,dp_y} = \text{const}$ & $\dfrac{dN}{dz\,dp_z} = \text{const}$

separately.

1ST DEFINITION:
EMITTANCE: USE TRACE-SPACE OF ALL PARTICLES IN A GIVEN SLICE OF BEAM.



INSTEAD OF p_x USE $x' = \dfrac{V_x}{V_z}$ (FOR NON-ACCELERATING PARAXIAL BEAM, x' PROPORTIONAL TO MOMENTUM)

EMITTANCE $\equiv \dfrac{1}{\pi} \cdot$ AREA OF SMALLEST ELLIPSE WHICH ENCLOSES ALL PARTICLES. (TRACE-SPACE DEFINITION)

(INTUITIVELY, PRODUCT OF WIDTH IN x, TIMES WIDTH IN x', SO IT IS ESSENTIALLY (WITHIN FACTOR OF π) = PHASE SPACE AREA OF BEAM.

2ND DEFINITION INVOLVES STATISTICAL AVERAGES OF 2ND ORDER QUANTITIES (SUCH AS rms).

$$\epsilon_x \equiv 4\left(\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right)^{1/2}$$

For an upright uniform beam (in phase space): $\langle x^2\rangle = \dfrac{r_x^2}{4}$ $\langle x'^2\rangle = \dfrac{x'^2_{max}}{4}$
& $\langle xx'\rangle = 0$

$$\Rightarrow \epsilon_x = r_x\, x'_{max} = \frac{\text{Area}}{\pi}$$

Normalized Emittance

For a beam that is accelerating, return to x, p_x as definition of phase space area:

$p_x = \gamma m v_x = \gamma m v_z x'$ Again, assuming $v \simeq v_z$

$$\Rightarrow \epsilon_{Nx} \equiv 4\gamma\beta\left(\langle x^2\rangle\langle x'^2\rangle - \langle x x'\rangle^2\right)^{1/2} = \gamma\beta\,\epsilon_x$$

$$= \frac{4}{m}\left(\langle x^2\rangle\langle p_x^2\rangle - \langle x p_x\rangle^2\right)^{1/2}$$

SINCE EMITTANCE IS THE AVERAGE PHASE SPACE AREA OF BEAM (AVERAGING OVER EMPTY SPACE) THE EMITTANCE IN GENERAL GROWS AS A BEAM FILAMENTS (ENGULFING EMPTY SPACE).

Brightness

THE DENSITY OF PARTICLES IN 6-D PHASE SPACE IS:

$$\frac{dN}{dx\,dy\,dz\,dp_x\,dp_y\,dp_z} = f \quad \longleftarrow \text{MICROSCOPIC DENSITY}$$

DEFINE A QUANTITY $\bar{f}$ WHICH IS THE PHASE-SPACE DENSITY IN AN AVERAGE SENSE

$$\bar{f} = \left\langle \frac{dN}{dx\,dp_x\,dy\,dp_y\,dz\,dp_z} \right\rangle = \frac{(I\Delta t)/q}{\pi^3 \epsilon_{Nx}\,\epsilon_{Ny}\,\epsilon_{Nz}}$$

Note $f(\underline{x},\underline{p})$ = constant along a trajectory, whereas $\bar{f}$ usually is a decreasing function of z.

$$\text{NORMALIZED BRIGHTNESS} \quad B_N \equiv \frac{I}{\epsilon_{Nx}\,\epsilon_{Ny}}$$

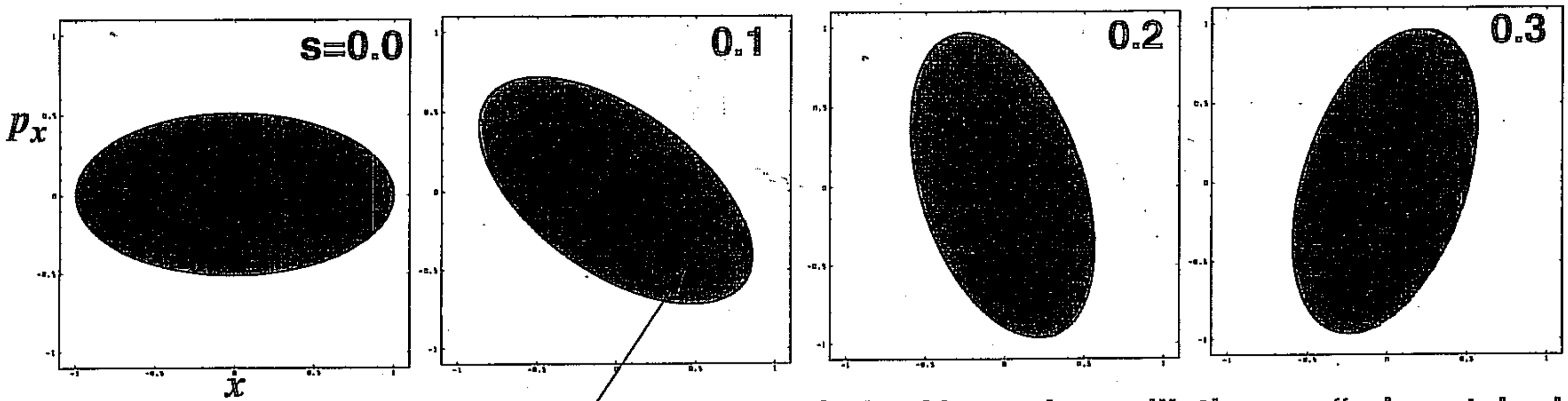
IS A USEFUL MEASURE OF 4D AVERAGE PHASE SPACE DENSITY, (if Δt = constant, & $\perp$,$\parallel$ motion is uncoupled.)

For non-accelerating beams the unnormalized brightness B (also if Δt = const. & $\perp$,$\parallel$ motion uncoupled):

$$\Rightarrow B \equiv \frac{I}{\epsilon_x\,\epsilon_y} \quad \text{MEASURES PHASE SPACE DENSITY.}$$

Emittance constant for linear force profile & matched beams

Linear force profile ($x'' = -k^2 x$) => Phase space area preserved, ellipse stays elliptical.



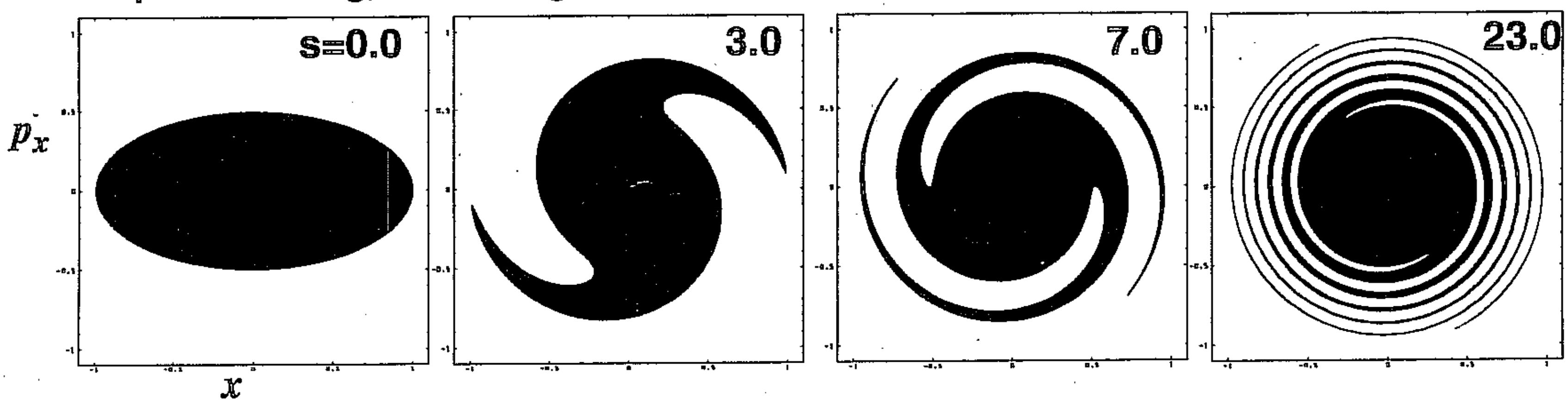
Emittance = phase space area

Emittance constant if forces linear

Here, width of beam is oscillating or "mismatched."

Non-linear forces (e.g. $x'' = -k^2 x + \varepsilon x^3$) $\Rightarrow$ position-dependent frequency

$\Rightarrow$ phase mixing, increasing effective area $\Rightarrow$ Emittance increases if forces non-linear



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